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Discrete symmetries are studied in warped space-times with one extra dimension. In particular, we analyze the compatibility of five- and four-dimensional charge conjugation, parity, time reversal and the orbifold symmetry  $Z_2$  with localization of fermions on the four-dimensional brane-world and Lorentz invariance. We then show that, when a suitable topological scalar field (the “kink”) is included, fermion localization is a consequence of (five-dimensional) CPT invariance.

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## I. INTRODUCTION

Recently there has been a revived interest in models containing extra spatial dimensions. The first time this idea was put in concrete form possibly dates back to the '20s [1], when the existence of more than four dimensions was employed in an attempt to unify gravity with the electromagnetic field. The lack of signals from (and loss into) the fifth dimension was assured by taking the size of the extra dimension small, which gives a huge mass gap between the ground state fields and the Kaluza-Klein (KK) modes. One could then Fourier-transform all fields, retain just the lowest mode and integrate out any dependence on the fifth dimension. In the context of such theories, the discrete symmetries of charge conjugation (C), parity (P) and time reversal (T) were investigated in Ref. [2], where it was found that the Dirac equation contains terms which violate CP and T. However, such violating terms are highly suppressed by factors of  $1/M_p$ , where  $M_p$  is the Planck mass, and are not likely to lead to any observable effect.

Higher dimensional spaces naturally arise in string theory [3], where one must compactify the fundamental theory down to our four-dimensional world. In Ref. [4] a compactification mechanism was built for the strongly coupled  $E_8 \times E_8$  heterotic string, whose low energy limit is eleven-dimensional supergravity. Six dimensions are then compactified on a Calabi-Yau manifold and integrated out to leave a five-dimensional (bulk) space-time bounded by two copies of the same D3-brane [5]. Matter particles are low energy excitations of open strings, whose end-points are tight to the D3-brane, which could thus represent our (brane-) world. Gravitons are instead closed strings and can propagate also in the extra direction, which has topology  $S/Z_2$ . Further, this construction yields a relation between the Planck scale on the D3-brane and the fundamental string scale which allows the latter to be much smaller than the former.

Starting from this result, several models have been proposed, with a variety in the number of extra dimensions, which can further be either compact [6] or infinitely extended but with a warp factor along the extra directions [7]. In both cases, the parameters can be tuned in such a

way that the fundamental mass scale  $M$  is small enough to lead to new physics slightly above 1 TeV without violating Newton's law at the present level of confidence [8]. One of the main concerns in such models is actually to provide an explicit, low energy, confining mechanism for the matter fields which does not violate any of the tested properties of the Standard Model and yields, at the same time, predictable effects to be probed by the forthcoming generation of detectors [6,9]. Early proposals for confining matter fields on a four-dimensional wall are actually older (see Refs. [10]) and make use of a mechanism which was proposed to generate masses dynamically by coupling fermions to a solitonic state (“kink”) of a scalar field. Such a mechanism yields a brane whose thickness [11,12] can then be related to the existence of KK-like partners of the SM particles and constrained by comparing with precision measurements such as the anomalous magnetic moment of leptons [13]. The fact that bulk gravitons living in the extra dimensions have not yet been detected is then generally a consequence of the small coupling with SM particles, namely  $1/M_p$  [6,14].

In the present paper we analyze general aspects of the discrete symmetries C, P and T for fermions interacting with gauge fields and the  $Z_2$  symmetry along the extra dimension in much the same spirit as such symmetries were studied in Ref. [2]. (For the sake of simplicity, we just consider one extra dimension and abelian gauge fields.) In particular, we are interested in the interplay between CPT (and Lorentz invariance) in five and four dimensions and the mechanism of confinement for fermions mentioned above from the viewpoint of local field theory, without attempting any rigorous connection to string theory or D-brane theory.

In the next Section we define the model. In Section III we introduce C, P and T both from the five- and four-dimensional points of view and highlight some ambiguities in the definition of the corresponding transformations. However, we show that there is a unique form of five-dimensional CPT invariance which is compatible with localization. Finally, in Section IV we summarize and comment on our results.

## II. LOCALIZED FERMIONS AND FERMION DOUBLING

We consider a five-dimensional space-time with (non-factorizable) metric  $g_{AB}$  ( $A, B = 0, \dots, 4$ ) given by

$$ds^2 = a^2 d\mathbf{x} \cdot d\mathbf{x} + dy^2, \quad (1)$$

where  $\cdot$  is the flat (Minkowski) scalar product between four-dimensional vectors. We do not assume for the warp factor  $a = a(y)$  any given form. In principle,  $a$  could be determined once the matter content is given by solving Einstein field equations, however, in this article we shall just need to know that it is  $Z_2$ -even, to wit  $a(-y) = a(y)$ . The four-dimensional metric on the brane, which is located at  $y = 0$ , is  $\boldsymbol{\eta} = \text{diag}[-1, +1, +1, +1]$  and the four brane coordinates are denoted by  $\mathbf{x} = (t, \vec{x})$ . In order to describe spinors, we introduce the ‘‘pentad’’

$$e^A_B = \text{diag} \left[ \frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \frac{1}{a}, 1 \right]. \quad (2)$$

All fields will be taken as functions of  $(\mathbf{x}, y)$  and Dirac matrices satisfy the five-dimensional Clifford algebra (note the appearance of the warp factor  $a$ )

$$e^A_C e^B_D \{ \gamma^C, \gamma^D \} = -2 g^{AB}, \quad (3)$$

and are given in the Weyl representation by

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \quad (4)$$

with  $\vec{\sigma}$  the Pauli spin matrices, and  $\gamma^4 = i\gamma^5 = i \text{diag}[+\mathbb{I}_2, -\mathbb{I}_2]$ , where  $\mathbb{I}_2$  is the  $2 \times 2$  identity matrix.

The five-dimensional action for one Dirac spinor  $\psi = \psi(\mathbf{x}, y)$  minimally coupled to a gauge boson  $A_A = A_A(\mathbf{x}, y)$  in the background with metric (1) is given by

$$S = \int a^4 dy \int d^4 x \bar{\psi} \left[ \frac{i}{a} \not{D} + i\gamma^4 (D_y + \omega) - V \right] \psi, \quad (5)$$

where  $D_y = \partial_y + i e A_4$ ,  $\not{D} = \not{\partial} + i e \not{A}$ ,  $\not{\partial} = \boldsymbol{\gamma} \cdot \boldsymbol{\partial}$ ,  $\not{A} = \boldsymbol{\gamma} \cdot \mathbf{A}$  and  $\omega = \partial_y \ln(a^2)$  is a sum of spin connection terms. In the following Dirac spinor fields are taken in the Weyl representation,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad (6)$$

with  $L$  for left- and  $R$  for right-handed, and their field equation reads

$$\left[ \frac{i}{a} \not{D} + i\gamma^4 (\partial_y + i e A_4 + \omega) - V \right] \psi = 0. \quad (7)$$

The potential  $V = m + \Phi$ , where the constant  $m$  is the five-dimensional mass of the spinor and  $\Phi = \Phi(\mathbf{x}, y)$  is a (complex) scalar field.

The (orbifold) symmetry  $Z_2 : y \rightarrow -y$  requires that the Lagrangian density appearing in the action for any given set of fields be an even function of  $y$ . If we then assume  $S$  be  $Z_2$  invariant, one finds that  $\psi_L$  and  $\psi_R$  must have opposite parity under  $y \rightarrow -y$  and the potential  $V$  must be  $Z_2$ -odd, that is

$$m = 0 \quad (8)$$

$$\Phi(\mathbf{x}, y) = -\Phi(\mathbf{x}, -y).$$

An example of scalar field  $\Phi$  which is used to confine fermions on a domain wall is that of the kink (+) or anti-kink (-) [10],

$$\Phi = \pm q \tanh(py). \quad (9)$$

The above (positive) parameters  $p$  and  $q$  are related to the thickness of the brane and energy threshold of confinement, as discussed in Ref. [11]. There it is shown that, on properly taking into account the finite thickness of the brane, the field  $\psi$  can be expanded in a tower of KK-like modes. Here, for simplicity, we consider infinitesimal the thickness of the brane and set  $p \rightarrow \infty$  in (9). This yields a scalar field

$$\Phi \sim \pm q \text{sgn}(y), \quad (10)$$

which confines just *one* of the chiralities of a Dirac spinor. This fact prevents a mass term for the localized spinor and, as a consequence, we cannot interpret the confined fermions as the usual standard model particles. This problem is also present in the case of a thick brane but just for the zero-mode of the KK tower (see Appendix A for details).

A solution to the above problem is the ‘‘fermion doubling’’: since we need both chiralities to recover the correct low energy phenomenology, one doubles the fermion fields  $\psi \rightarrow (\psi_1, \psi_2)$  and couples  $\psi_1$  to the kink [plus sign in Eq. (9)] and  $\psi_2$  to the ‘‘anti-kink’’ [minus sign in Eq. (9)]. It is useful to organize the two kind of fermions in a single field [15]

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \begin{pmatrix} \psi_{1,L} \\ \psi_{1,R} \\ \psi_{2,L} \\ \psi_{2,R} \end{pmatrix}, \quad (11)$$

where  $\psi_1$  and  $\psi_2$  are two four-component spinors living in five dimensions. Further, because of the  $Z_2$  symmetry,  $\psi_{1,L}$  and  $\psi_{2,R}$  must be even (odd) functions of  $y$ , while  $\psi_{1,R}$  and  $\psi_{2,L}$  must be odd (even). After the fermion doubling, the five-dimensional action (5) can be written as

$$S_{(5)} = \int a^4 dy \int d^4 x \bar{\Psi} \left\{ \frac{i}{a} [\boldsymbol{\gamma} \otimes \mathbb{I}_2] \cdot \mathbf{D} + i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - \Phi [\mathbb{I}_4 \otimes \sigma^3] - \frac{\mu}{a} [\mathbb{I}_4 \otimes \sigma^1] \right\} \Psi, \quad (12)$$

where we have denoted by  $\mathbb{I}_4$  the  $4 \times 4$  identity matrix. The last term is an interaction term between the two families ( $\psi_1$  and  $\psi_2$ ) of fermions which can give rise to a four-dimensional mass. In fact, on factorizing the spinor  $\psi_i = \alpha(y) f_i(\mathbf{x})$  and first imposing the equation for the localization,

$$(D_y + \omega + \Phi) \alpha = 0 , \quad (13)$$

and then the vanishing condition for half of the components,

$$f_{2,L} = f_{1,R} = 0 , \quad (14)$$

one obtains the usual four-dimensional action (multiplied by a numerical coefficient)

$$S_{(4)} = \int |\alpha|^2 a^3 dy \int d^4x \bar{\chi} [i \not{D} - \mu] \chi . \quad (15)$$

for the spinor field

$$\chi = \begin{pmatrix} f_{1,L} \\ f_{2,R} \end{pmatrix} \quad (16)$$

of mass  $\mu$ . Clearly the above reduction is successful if the integration on  $y$  is convergent. Note that the spinor so obtained ( $\chi$ ) is organized in such a way that its  $L$  component is the  $L$  component of the family number one and its  $R$  component is the  $R$  component of the family number two. Note also that we have considered the gauge field  $A_A$  as independent of  $y$ .

From the variation of the action in Eq. (12) we obtain the following equation of motion

$$\left\{ \frac{i}{a} [\gamma \otimes \mathbb{I}_2] \cdot \mathbf{D} + i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - \Phi [\mathbb{I}_4 \otimes \sigma^3] - \frac{\mu}{a} [\mathbb{I}_4 \otimes \sigma^1] \right\} \Psi = 0 . \quad (17)$$

The gauge field  $A_A$  [16] couples minimally to the fermions via the “electric” charge  $e$ . We shall not need to work out the field equations satisfied by  $A_A$ , so that in the following it can be taken as a generic external field. Of course, from the usual parity properties of the electromagnetic tensor  $F_{AB} = \partial_A A_B - \partial_B A_A$ , one obtains that the vector potential must satisfy the following relations

$$\begin{aligned} A_0(t, \vec{x}, y) &= A_0(-t, \vec{x}, y) \\ &= A_0(t, -\vec{x}, y) \\ &= A_0(t, \vec{x}, -y) \\ \vec{A}(t, \vec{x}, y) &= -\vec{A}(-t, \vec{x}, y) \\ &= -\vec{A}(t, -\vec{x}, y) \\ &= \vec{A}(t, \vec{x}, -y) \\ A_4(t, \vec{x}, y) &= -A_4(-t, \vec{x}, y) \\ &= A_4(t, -\vec{x}, y) \\ &= -A_4(t, \vec{x}, -y) . \end{aligned} \quad (18)$$

### III. DISCRETE SYMMETRIES

We are now ready to investigate the discrete symmetries  $Z_2$ , C, P and T. As it will be clear from the following, such symmetries are no more uniquely represented when a fifth dimension is introduced. In particular, since we are dealing with an effective five-dimensional field theory, one could define discrete symmetries from the purely five-dimensional point of view (see Section III A) and consider that CPT holds in five dimensions, regardless of fermion doubling and thus for each family  $\psi_i$  separately. However, in so doing, one obtains something which does not relate to the four-dimensional description in a *a priori* clear way, since the fermion doubling mixes components of the two families, and new definitions of C, P, and T are needed (Section III B).

It is our main aim to clarify the compatibility of the above two aspects and their relation with localization in general. For our discussion, the precise shape of  $a$  and  $\Phi$  in the bulk is not required and we shall mostly need to know just their parity properties.

#### A. Five-dimensional symmetries

In this Section we define discrete symmetries as acting on each one of the two families  $\psi_i$  separately. These are expected to be the fundamental symmetries of the five-dimensional effective field theory. Having grouped together the two families  $\psi_i$  according to Eq. (11), we now have to deal with eight-spinors  $\Psi$ . Correspondingly, the operators generating (local) Lorentz transformations as well as C, P and T will be represented by  $8 \times 8$  matrices of the form  $F \otimes \mathbb{I}_2$ , where  $F$  is  $4 \times 4$ . We shall just consider candidates of C, P and T which have previously appeared in the literature.

##### 1. Charge conjugation

The first possible definition of five-dimensional charge conjugation ( $C_5$ ) which we consider is the usual charge conjugation (C) [17] acting on each family  $\psi_i$ ,

$$C_5 [\Psi] (x^A) = [C \otimes \mathbb{I}_2] \Psi^*(x^A) , \quad (19)$$

where  $C = i \gamma^2$  and satisfies

$$\begin{aligned} C(\gamma^\mu)^* &= -\gamma^\mu C \\ C(\gamma^4)^* &= \gamma^4 C . \end{aligned} \quad (20)$$

Invariance of Eq. (17) requires

$$\{ i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - [\mathbb{I}_4 \otimes \sigma^3] \text{Im}(\Phi) \} \Psi = 0 . \quad (21)$$

If we choose a scalar field of the type in Eq. (9), we do not have compatibility with Eq. (13) for localization of

spinors because the kink is a real function. It then follows that localization breaks  $C_5$ . However, if  $\Phi$  is not real, the above condition (21) is not invariant under rotations and boosts involving the fifth dimension, therefore it breaks five-dimensional Lorentz invariance (but is still compatible with four-dimensional Lorentz symmetry).

We can alternatively define charge conjugation in the following way [18]:

$$\tilde{C}_5 [\Psi] (x^A) = [\tilde{C} \otimes \mathbb{I}_2] \Psi^*(x^A) \quad (22)$$

where  $\tilde{C} = i \gamma^2 \gamma^5$  and satisfies

$$\tilde{C} (\gamma^A)^* = \gamma^A \tilde{C} . \quad (23)$$

Invariance of Eq. (17) requires

$$\left\{ \text{Re}(\Phi) [\mathbb{I}_4 \otimes \sigma^3] + \frac{\mu}{a} [\mathbb{I}_4 \otimes \sigma^1] \right\} \Psi = 0 . \quad (24)$$

This is satisfied for every  $\Psi$  if

$$\begin{cases} \mu = 0 \\ \text{Re}(\Phi) = 0 . \end{cases} \quad (25)$$

The second condition is again in contrast with the solitonic state (9), but (24) is compatible with Lorentz invariance in five and four dimensions.

There is also a third possible definition which involves a change of sign of the coordinate  $y$  [2]:

$$C_{5(y)} [\Psi] (\mathbf{x}, y) = [C \otimes \mathbb{I}_2] \Psi^*(\mathbf{x}, -y) . \quad (26)$$

Invariance of Eq. (17) now requires

$$\Phi(\mathbf{x}, y) = \Phi^*(\mathbf{x}, -y) , \quad (27)$$

and is not compatible with the  $Z_2$  symmetry if we take  $\Phi$  real.

To summarize, there appear to be no known definition of charge conjugation in five dimensions which is compatible with localization given by the use of the scalar field in Eq. (9).

## 2. Parity

We define the five-dimensional parity ( $P_5$ ) as the usual parity ( $P$ ) acting on each family  $\psi_i$ ,

$$P_5 [\Psi] (t, \vec{x}, y) = [P \otimes \mathbb{I}_2] \Psi(t, -\vec{x}, -y) , \quad (28)$$

where  $P = i \gamma^0$ . Invariance of Eq. (17) requires

$$\Phi(t, \vec{x}, y) = \Phi(t, -\vec{x}, -y) . \quad (29)$$

This is in contrast with Eq. (9) so that the five-dimensional parity is broken if we choose  $\Phi$  kink-like, and the conclusion is the same as for charge conjugation.

We define the five-dimensional time reversal ( $T_5$ ) as the usual time reversal ( $T$ ) acting on each family  $\psi_i$ ,

$$T_5 [\Psi] (t, \vec{x}, y) = [T \otimes \mathbb{I}_2] \Psi^*(-t, \vec{x}, y) , \quad (30)$$

where  $T = i \gamma^1 \gamma^3$ . Invariance of Eq. (17) requires

$$\Phi(t, \vec{x}, y) = \Phi^*(-t, \vec{x}, y) . \quad (31)$$

If  $\Phi$  is given by Eq. (9), this symmetry is satisfied.

## 4. CPT

One expects that, if the five-dimensional theory is a genuine (locally Lorentz invariant) field theory, CPT holds [19]. However, we have three ways of defining CPT depending on the choice of the operator of charge conjugation and we would like to pick up the one which is compatible with the localization mechanism given by the soliton in Eq. (9) and Eq. (13).

The preferred CPT is thus given by the combination  $C_5 P_5 T_5$ . In fact, invariance of Eq. (17) under the transformation

$$(C_5 P_5 T_5) [\Psi] (x^A) = [\gamma^5 \otimes \mathbb{I}_2] \Psi^*(-x^A) , \quad (32)$$

requires:

$$\begin{aligned} & \left\{ i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) \right. \\ & \left. - [\mathbb{I}_4 \otimes \sigma^3] \frac{\Phi(x^A) - \Phi(-x^A)}{2} \right\} \Psi = 0 . \end{aligned} \quad (33)$$

If  $\Phi$  is kink-like the latter equation becomes precisely Eq. (13) for the localization. Thus we conclude that localization on the brane is a consequence of CPT invariance of the (effective) five-dimensional field theory. However, since the condition (33) suffers of the same problem as Eq. (21), it violates (local) five-dimensional Lorentz invariance (while preserving four-dimensional Lorentz invariance). This is not surprising, since confined states are not invariant under rotations and boosts which involve the fifth dimension or, put another way, the brane itself represents a preferred frame. Note that separately  $C_5$  and  $P_5$  are broken but together with  $T_5$  they give Eq. (33) that becomes (13) when (9) [and (14) for zero modes] are imposed.

For completeness we also consider the remaining cases. In particular, invariance of Eq. (17) under the combination  $\tilde{C}_5 P_5 T_5$ ,

$$(\tilde{C}_5 P_5 T_5) [\Psi] (x^A) = -[\mathbb{I}_4 \otimes \mathbb{I}_2] \Psi^*(-x^A) , \quad (34)$$

requires

$$\left\{ \left[ \mathbb{I}_4 \otimes \sigma^3 \right] \frac{\Phi(x^A) + \Phi(-x^A)}{2} + \frac{\mu}{a} \left[ \mathbb{I}_4 \otimes \sigma^1 \right] \right\} \Psi = 0 . \quad (35)$$

If we take  $\Phi$  to be given as in Eq. (9) we have to set  $\mu = 0$ . This confirms that  $\tilde{C}_5$  is a bad definition of charge conjugation if we want massive fermions on the brane, although this choice is fully compatible with five-dimensional Lorentz symmetry.

Finally, invariance of Eq. (17) under the combination which includes  $C_{5(y)}$ ,

$$(C_{5(y)} P_5 T_5) [\Psi](\mathbf{x}, y) = [\gamma^5 \otimes \mathbb{I}_2] \Psi^*(-\mathbf{x}, y) , \quad (36)$$

requires

$$\Phi(\mathbf{x}, y) = \Phi(-\mathbf{x}, y) . \quad (37)$$

This constraint is satisfied by Eq. (9). We will see below that this definition of CPT is really four-dimensional.

### 5. CP and T

Since  $T_5$  holds for the choice in Eq. (9), one expects that CP is unbroken as a consequence of CPT invariance. We have three different ways of defining CP depending on the three possible choices of charge conjugation. Invariance of Eq. (17) under  $C_5$  and  $P_5$ ,

$$(C_5 P_5) [\Psi](x^A) = - [\gamma^2 \gamma^0 \otimes \mathbb{I}_2] \Psi^*(t, -\vec{x}, -y) , \quad (38)$$

requires:

$$\left\{ i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - [\mathbb{I}_4 \otimes \sigma^3] \frac{\Phi(t, \vec{x}, y) - \Phi^*(t, -\vec{x}, -y)}{2} \right\} \Psi = 0 . \quad (39)$$

If  $\Phi$  is given by Eq. (9) the above condition again becomes the equation for localization and CP is unbroken for fermions living on the brane (while five-dimensional Lorentz invariance is violated).

The second possible definition makes use of  $\tilde{C}_5$  and  $P_5$ :

$$(\tilde{C}_5 P_5) [\Psi](x^A) = - [\gamma^2 \gamma^5 \gamma^0 \otimes \mathbb{I}_2] \Psi^*(t, -\vec{x}, -y) . \quad (40)$$

Invariance of Eq. (17) then requires:

$$\left\{ [\mathbb{I}_4 \otimes \sigma^3] \frac{1}{2} (\Phi(t, \vec{x}, y) + \Phi^*(t, -\vec{x}, -y)) + \frac{\mu}{a} [\mathbb{I}_4 \otimes \sigma^1] \right\} \Psi = 0 . \quad (41)$$

Note that if  $\Phi$  is kink-like the first term vanishes and the above condition is verified for every  $\Psi$  if  $\mu = 0$ . We

conclude that  $\tilde{C}_5 P_5$  respects five-dimensional Lorentz invariance and is satisfied if there is no interaction term between the two families of fermions. However, as we mentioned, this term is necessary to obtain the four-dimensional mass term and to recover the low energy phenomenology.

We can also define CP using  $C_{5(y)}$  and  $P_5$ :

$$(C_{5(y)} P_5) [\Psi](x^A) = - [\gamma^2 \gamma^0 \otimes \mathbb{I}_2] \Psi^*(t, -\vec{x}, y) . \quad (42)$$

In this case invariance of Eq. (17) yields:

$$\Phi(t, \vec{x}, y) = \Phi^*(t, -\vec{x}, y) . \quad (43)$$

This condition is satisfied if  $\Phi$  is given by Eq. (9) and, as we show below, coincides with the condition that ensures CP in four dimensions.

## B. Four-dimensional symmetries

In this Section we define discrete symmetries as acting on eight-spinors as seen from the four-dimensional point of view. In other words we are looking for operators that act on  $\psi_{1,L}$  and  $\psi_{2,R}$  ( $\psi_{2,L}$  and  $\psi_{1,R}$ ) as if they belonged to the same Dirac spinor  $\psi^{(1)}$  ( $\psi^{(2)}$ ) and are not substantially affected by the existence of a fifth coordinate, hence they must be represented by matrices of the form

$$\begin{bmatrix} F_{2 \times 2}^{(1)} & 0_{2 \times 4} & F_{2 \times 2}^{(1)} \\ 0_{4 \times 2} & F_{4 \times 4}^{(2)} & 0_{4 \times 2} \\ F_{2 \times 2}^{(1)} & 0_{2 \times 4} & F_{2 \times 2}^{(1)} \end{bmatrix} , \quad (44)$$

in which upper indices denote blocks acting on the corresponding spinor  $\psi^{(i)}$ .

Strictly speaking such symmetries are not as fundamental as the five-dimensional analogues and are properly defined only for states localized on the brane. Further, they all naively respect four-dimensional Lorentz invariance, since four-dimensional Lorentz transformations must be represented by matrices of the form (44) and, as we show below, C, P and T coincide with the standard four-dimensional expressions. We finally note that four-dimensional Lorentz invariance for confined states also follows from the effective action (15).

### 1. Charge conjugation

We define the four-dimensional charge conjugation as

$$C_4 [\Psi](x^A) = [C \otimes \sigma^1] \Psi^*(x^A) \quad (45)$$

where, as before,  $C = i \gamma^2$ . Invariance of Eq. (17) now requires

$$\{ i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - [\mathbb{I}_4 \otimes \sigma^3] \text{Re}(\Phi) \} \Psi = 0 . \quad (46)$$

If we choose (9), this symmetry is satisfied if the equation for localization (13) holds.

As in five dimensions, this is not the only way to define this operation. In fact we can also define the four-dimensional charge conjugation as

$$\tilde{C}_4 [\Psi] (x^A) = [\tilde{C} \otimes \sigma^1] \Psi^*(x^A) \quad (47)$$

where, as before,  $\tilde{C} = i\gamma^2\gamma^5$ . Invariance of Eq. (17) requires Eq. (24) and we obtain the same results as in five dimensions [see Eq. (25)].

Analogously to the five-dimensional case, we could also define charge conjugation as

$$C_{4(y)} [\Psi] (\mathbf{x}, y) = [C \otimes \sigma^1] \Psi^*(\mathbf{x}, -y) . \quad (48)$$

Invariance of Eq. (17) then requires (27).

To conclude, while  $C_4$ -invariance is compatible with localization, both  $\tilde{C}_4$  and  $C_{4(y)}$  lead to the same conditions as the analogous  $\tilde{C}_5$  and  $C_{5(y)}$  and are therefore in contrast with localization. If one wants an operator of charge conjugation which, from a four-dimensional point of view, gives rise to an invariance of Eq. (17), one is then forced to discard  $\tilde{C}_4$  and  $C_{4(y)}$  as candidates for charge conjugation.

### 2. Parity

We define the four-dimensional parity as follows

$$P_4 [\Psi] (t, \vec{x}, y) = [P \otimes \sigma^1] \Psi(t, -\vec{x}, y) , \quad (49)$$

where, as before,  $P = i\gamma^0$ . Invariance of Eq. (17) requires

$$\left\{ i [\gamma^4 \otimes \mathbb{I}_2] (D_y + \omega) - [\mathbb{I}_4 \otimes \sigma^3] \frac{\Phi(t, \vec{x}, y) + \Phi(t, -\vec{x}, y)}{2} \right\} \Psi = 0 . \quad (50)$$

One more time, if  $\Phi$  is given by (9), we get that  $P$  is satisfied when the equation for localization holds.

### 3. Time reversal

The four-dimensional definition of this operation  $T_4$  is exactly the same as  $T_5$  in five dimensions.

### 4. CPT

Invariance of Eq. (17) under the combination

$$\begin{aligned} (C_4 P_4 T_4) [\Psi] (\mathbf{x}, y) &= [\gamma^5 \otimes \mathbb{I}_2] \Psi^*(-\mathbf{x}, y) \\ &= (C_{5(y)} P_5 T_5) [\Psi] (\mathbf{x}, y) \end{aligned} \quad (51)$$

requires Eq. (37). If  $\Phi$  does not depend on  $\mathbf{x}$  (as the kink) the latter equation is automatically satisfied.

### 5. CP and T

Invariance of Eq. (17) under combination of the previous operations  $C_4$  and  $P_4$ ,

$$\begin{aligned} (C_4 P_4) [\Psi] (x^A) &= - [\gamma^2 \gamma^0 \otimes \mathbb{I}_2] \Psi^*(t, -\vec{x}, y) \\ &= (C_{5(y)} P_5) [\Psi] (x^A) , \end{aligned} \quad (52)$$

requires Eq. (43).

## IV. CONCLUSIONS

We have analyzed discrete symmetries in a four-dimensional brane-world of codimension one and their relation with localization and Lorentz invariance. We found that the usual way of localizing fermions with a (anti-) kink [10] is compatible with CPT in five dimensions (and follows from it), provided one uses  $C_5$ ,  $P_5$  and  $T_5$  given respectively in Eqs. (19), (28) and (30). In fact, we have shown that, among the three expressions of CPT that we have considered in five dimensions, just  $C_5 P_5 T_5$  is consistent with a brane-world. On the contrary,  $\tilde{C}_5 P_5 T_5$  is not compatible with a mass term [such as the vacuum expectation value of the Higgs field,  $\mu$  in Eq. (12)] for fermions on the brane and  $C_{5(y)} P_5 T_5$  is actually four-dimensional (since it equals  $C_4 P_4 T_4$ ). Correspondingly, CPT in four dimensions (*i.e.*,  $C_4 P_4 T_4$ , which is properly defined only for localized fermions) does not impose any further restriction except Eq. (37). It is also possible to obtain Eq. (13) for the localization from a purely four-dimensional point of view: it is sufficient to impose the invariance of the Dirac equation (17) under either  $C_4$  or  $P_4$ .

It is nevertheless worth noting that it is  $\tilde{C}_5 P_5 T_5$  which respects five-dimensional (local) Lorentz invariance. Instead,  $C_5 P_5 T_5$  breaks this symmetry (while preserving Lorentz invariance on the brane), as one could naively expect since the brane represents a preferred frame. Since Lorentz invariance is a main hypothesis for the CPT theorem (in any dimensions) [19], this seems to play in favour of the expression involving  $\tilde{C}_5$  and one would therefore expect that five-dimensional CPT is broken when there are massive fermions on the brane. Whether the breakdown of this five-dimensional symmetry leads to observable effects in our (brane-)world will be the subject of subsequent investigations.

Since the origin of the reference frame on the brane is arbitrary, the condition (37) for CPT invariance in four dimensions implies that the kink must be static and homogeneous, although such a condition is not required for CPT invariance from the five-dimensional point of view. One would expect that background fields, such as  $\Phi$ , are not static in a cosmological context (at least during the early stages of the Universe) and can therefore be a source of CPT asymmetries on the brane-world. Analogously, spatial inhomogeneities could generate local violations of

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## APPENDIX A: THICK BRANE

In this Section we explicitly consider the case of a thick brane. For this purpose it is useful to approximate the kink as [11,13]

$$\Phi = \begin{cases} -(m_f^2/2) L_f & y < -L_f \\ (m_f^2/2) y & |y| < L_f \\ +(m_f^2/2) L_f & y > +L_f \end{cases} \quad (\text{A1})$$

where  $L_f$  is the typical brane thickness for fermions, and take  $a(y) \sim 1$  for  $|y| < L_f$ . It is then easy to show that fermions have a confined massless (chiral) mode, together with a tower of states (which are allowed only if their mass is smaller than the threshold  $m_f^2 L_f/2$  [13]). Since  $\gamma^5 = \Pi_L - \Pi_R$  (the difference between L and R chiral projectors), one can introduce “creation and annihilation” operators

$$\hat{a}^\dagger = -\frac{1}{m_f} (\partial_y - \Phi) , \quad \hat{a} = \frac{1}{m_f} (\partial_y + \Phi) , \quad (\text{A2})$$

such that  $[\hat{a}, \hat{a}^\dagger] = 1$  and the Lagrangian density for  $|y| < L_f$  becomes

$$L_{(5)} = \bar{\Psi} (i \not{\partial} - m_f \hat{a} \Pi_L - m_f \hat{a}^\dagger \Pi_R) \Psi . \quad (\text{A3})$$

This allows an expansion for the fermions

$$\Psi(\mathbf{x}, y) = H_0(y) \Pi_L \psi^{(0)}(\mathbf{x}) + \sum_{n=1}^{N_f} [H_n(y) \Pi_L + H_{n-1}(y) \Pi_R] \psi^{(n)}(\mathbf{x}) , \quad (\text{A4})$$

where  $H_n$  are the normalized eigenfunctions of the harmonic oscillator. From the above it is clear that while the zero mode is a two-component spinor, modes with  $n \geq 1$  can have both chiralities.

Since the zero mode is massless (because  $\hat{a} H_0 = 0$ ),

$$(\not{\partial} + m_f \hat{a}) H_0 \Pi_L \psi^{(0)} = H_0 \not{\partial} \Pi_L \psi^{(0)} = 0 , \quad (\text{A5})$$

$\psi^{(0)}$  can be taken as a L Weyl spinor,  $\psi^{(0)} = \Pi_L \psi^{(0)}$ . We note in passing that  $\hat{a} H_0 = 0$  is precisely the equation which ensures the confinement of the left zero mode within a width  $\ell_f \sim 1/m_f$  around  $y = 0$ . Since for the (would-be) right zero mode the corresponding equation  $\hat{a}^\dagger \bar{R}_0 = 0$  does not admit any (non-vanishing) normalizable solution in  $y \in \mathbb{R}$ , we have set  $\Pi_R \psi^{(0)} = 0$ .

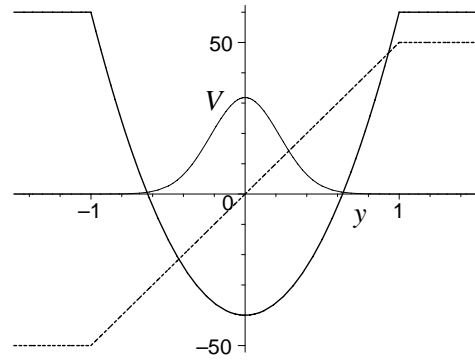


FIG. 1. Sketch of the scalar field  $\Phi$  (magnified by a factor of 5; dashed line) and the corresponding confining potential in Eq. (A6) (continuous line) for  $m_f^2/2 = 10$  (in units with  $L = 1$ ). The Gaussian curve represents the ground state  $H_0$  (magnified by a factor of 10).

The sum in Eq. (A4) ends with a maximum integer  $N_f < \infty$ . The reason for such a cut-off can be easily understood if we set  $\mathcal{A} = 0$  and write down the Klein-Gordon equation corresponding to the Dirac equation obtained from  $S$ ,

$$\begin{aligned} & [\not{\partial} - m_f (\hat{a}^\dagger \Pi_L - \hat{a} \Pi_R)] [\not{\partial} + m_f (\hat{a} \Pi_L + \hat{a}^\dagger \Pi_R)] \Psi \\ &= -[p^2 + m_f^2 (\hat{a}^\dagger \hat{a} \Pi_L + \hat{a} \hat{a}^\dagger \Pi_R)] \Psi \\ &= -[p^2 + (-\partial_y^2 + \Phi^2)] \Psi = 0 . \end{aligned} \quad (\text{A6})$$

It is thus clear that only those modes  $\psi_n$  whose eigenvalues  $m_f^2 n < \Phi^2(L) \equiv M_f^2$  can be retained (see Fig. 1).

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